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Masses of nS -wave Heavy Quarkonium Levels from QCD Sum Rules V.V.Kiselev

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Abstract

Using a specific scheme of the QCD sum rules, one derives an universal formula for the mass differences for the nS -levels of heavy quarkonium. This relation does not depend on the flavours of the heavy quarks, composing the quarkonium, and it is in a good agreement with the experimental mass values for the ψ - and Υ -families.

Introduction

The QCD sum rules [1] are one of the powerful tools for a description of nonperturbative characteristics of the heavy quark bound states. However, in the consideration of masses for basic states of heavy quarkonia, the QCD sum rule predictions have the accuracy, that is one order of magnitude lower than the accuracy of phenomenological potential models [2]. This is caused by that, first, the QCD sum rule calculations are made in a finite order of the QCD perturbation theory for the Wilson's coefficients and with a restricted set of the quark-gluon condensates, so that the results depend on an external unphysical parameter, defining a sum rule scheme for an averaging (the number of moment for the spectral density of current correlator or the Borel transform parameter). Second, one has an uncertainty in a value of the threshold s_{th} , discriminating the resonant region and the hadronic continuum. Moreover, the rapidly-dropping weight functions, defining the sum rule scheme, do not allow one to extract an information on the contribution by the higher excitations in the quarkonium, so that this contribution is obviously neglected.

Recently in refs.[3, 4, 5], one has offered the QCD sum rule scheme, which allows one to get the scaling relation for the leptonic constants f of the S -states of different quarkonia with the mass M and the reduced quark mass μ ,

$$\frac{f^2}{M} \left(\frac{M}{4\mu} \right)^2 = \text{const.} , \quad (1)$$

and the relation for the leptonic constants f_n of excited nS -states of the quarkonium

$$\frac{f_{n_1}^2}{f_{n_2}^2} = \frac{n_2}{n_1} , \quad (2)$$

independently of the flavours of heavy quarks, composing the quarkonium.

In the present paper, in the framework of the QCD sum rule scheme, offered in refs.[3, 4, 5], we derive the relation for the mass differences of nS -wave levels in the heavy quarkonium

$$\frac{M_n - M_1}{M_2 - M_1} = \frac{\ln n}{\ln 2} , \quad n \geq 2 , \quad (3)$$

independently of the heavy quark flavours.

In Section 1 we describe the used scheme of QCD sum rules and derive relation (3). In Section 2 we make the phenomenological analysis of relation (3), which is in a good agreement with the experimental ratios of mass differences in the ψ - and Υ -families. In the Conclusion the obtained results are summarized.

1 Heavy Quarkonium Sum Rules

Let us consider the two-point correlator functions of quark currents

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0 | T J_\mu(x) J_\nu^\dagger(0) | 0 \rangle , \quad (4)$$

$$\Pi_P(q^2) = i \int d^4x e^{iqx} \langle 0 | T J_5(x) J_5^\dagger(0) | 0 \rangle , \quad (5)$$

where

$$J_\mu(x) = \bar{Q}_1(x) \gamma_\mu Q_2(x) , \quad (6)$$

$$J_5(x) = \bar{Q}_1(x) \gamma_5 Q_2(x) , \quad (7)$$

$$(8)$$

Q_i is the spinor field of the heavy quark with $i = c, b$.

Further, write down

$$\Pi_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \Pi_V(q^2) + \frac{q_\mu q_\nu}{q^2} \Pi_S(q^2) , \quad (9)$$

where Π_V and Π_S are the vector and scalar correlator functions, respectively. In what follows we will consider the vector and pseudoscalar correlators: $\Pi_V(q^2)$ and $\Pi_P(q^2)$.

Define the leptonic constants f_V and f_P

$$\langle 0 | J_\mu(x) | V(\lambda) \rangle = i \epsilon_\mu^{(\lambda)} f_V M_V e^{ikx} , \quad (10)$$

$$\langle 0 | J_{5\mu}(x) | P \rangle = i k_\mu f_P e^{ikx} , \quad (11)$$

where

$$J_{5\mu}(x) = \bar{Q}_1(x) \gamma_5 \gamma_\mu Q_2(x) , \quad (12)$$

so that

$$\langle 0 | J_5(x) | P \rangle = i \frac{f_P M_P^2}{m_1 + m_2} e^{ikx} , \quad (13)$$

where $|V\rangle$ and $|P\rangle$ are the state vectors of 1^- and 0^- quarkonia, and λ is the vector quarkonium polarization, k is 4-momentum of the meson, $k_{P,V}^2 = M_{P,V}^2$.

Considering the charmonium ($\psi, \psi' \dots$) and bottomonium ($\Upsilon, \Upsilon', \Upsilon'' \dots$), one can easily show that the relation between the width of leptonic decay $V \rightarrow e^+ e^-$ and f_V has the form

$$\Gamma(V \rightarrow e^+ e^-) = \frac{4\pi}{9} e_i^2 \alpha_{em}^2 \frac{f_V^2}{M_V} , \quad (14)$$

where e_i is the electric charge of quark i .

In the region of narrow nonoverlapping resonances, it follows from eqs.(4) - (13) that

$$\frac{1}{\pi} \Im m \Pi_V^{(res)}(q^2) = \sum_n f_{Vn}^2 M_{Vn}^2 \delta(q^2 - M_{Vn}^2) , \quad (15)$$

$$\frac{1}{\pi} \Im m \Pi_P^{(res)}(q^2) = \sum_n f_{Pn}^2 M_{Pn}^4 \frac{1}{(m_1 + m_2)^2} \delta(q^2 - M_{Pn}^2) . \quad (16)$$

Thus, for the observed spectral function one has

$$\frac{1}{\pi} \Im m \Pi_{V,P}^{(had)}(q^2) = \frac{1}{\pi} \Im m \Pi_{V,P}^{(res)}(q^2) + \rho_{V,P}(q^2, \mu_{V,P}^2), \quad (17)$$

where $\rho(q^2, \mu^2)$ is the continuum contribution, which is not equal to zero at $q^2 > \mu^2$.

Moreover, the operator product expansion gives

$$\Pi^{(QCD)}(q^2) = \Pi^{(pert)}(q^2) + C_G(q^2) \langle \frac{\alpha_S}{\pi} G^2 \rangle + C_i(q^2) \langle m_i \bar{Q}_i Q_i \rangle + \dots, \quad (18)$$

where the perturbative contribution $\Pi^{(pert)}(q^2)$ is labeled, and the nonperturbative one is expressed in the form of sum of quark-gluon condensates with the Wilson's coefficients, which can be calculated in the QCD perturbative theory.

In eq.(18) we have been restricted by the contribution of vacuum expectation values for the operators with dimension $d = 4$. For $C_G^{(P)}(q^2)$ one has, for instance, [1]

$$C_G^{(P)} = \frac{1}{192m_1m_2} \frac{q^2}{\bar{q}^2} \left(\frac{3(3v^2 + 1)(1 - v^2)^2}{2v^5} \ln \frac{1 + v}{1 - v} - \frac{9v^4 + 4v^2 + 3}{v^4} \right), \quad (19)$$

where

$$\bar{q}^2 = q^2 - (m_1 - m_2)^2, \quad v^2 = 1 - \frac{4m_1m_2}{\bar{q}^2}. \quad (20)$$

The analogous formulae for other Wilson's coefficients can be found in Ref.[1]. In what follows it will be clear that the explicit form of coefficients has no significant meaning for the present consideration.

In the leading order of QCD perturbation theory it has been found for the imaginary part of correlator that [1]

$$\begin{aligned} \Im m \Pi_V^{(pert)}(q^2) &= \frac{\tilde{s}}{8\pi s^2} (3\bar{s}s - \bar{s}^2 + 6m_1m_2s - 2m_2^2s) \theta(s - (m_1 + m_2)^2) \\ \Im m \Pi_P^{(pert)}(q^2) &= \frac{3\tilde{s}}{8\pi s^2} (s - (m_1 - m_2)^2) \theta(s - (m_1 + m_2)^2), \end{aligned} \quad (22)$$

where $\bar{s} = s - m_1^2 + m_2^2$, $\tilde{s}^2 = \bar{s}^2 - 4m_2^2s$.

The one-loop contribution into $\Im m \Pi(q^2)$ can be included into the consideration (see, for example, Ref.[1]). However, we note that the more essential

correction is that of summing a set over the powers of (α_s/v) , where v is defined in eq.(20) and is a relative quark velocity, and α_s is the QCD interaction constant. In Ref.[1] it has been shown that account of the Coulomb-like gluonic interaction between the quarks leads to the factor

$$F(v) = \frac{4\pi}{3} \frac{\alpha_s}{v} \frac{1}{1 - \exp(-\frac{4\pi\alpha_s}{3v})} , \quad (23)$$

so that the expansion of the $F(v)$ over $\alpha_s/v \ll 1$ restores, precisely, the one-loop $O(\frac{\alpha_s}{v})$ correction

$$F(v) \approx 1 - \frac{2\pi}{3} \frac{\alpha_s}{v} \dots \quad (24)$$

In accordance with the dispersion relation one has the QCD sum rules, which state that, in average, it is true that, at least, at $q^2 < 0$

$$\frac{1}{\pi} \int \frac{\Im m \Pi^{(had)}(s)}{s - q^2} ds = \Pi^{(QCD)}(q^2) , \quad (25)$$

where the necessary subtractions are omitted. $\Im m \Pi^{(had)}(q^2)$ and $\Pi^{(QCD)}(q^2)$ are defined by eqs.(15) - (17) and eqs.(18) - (24), respectively. eq.(25) is the base to develop the sum rule approach in the forms of the correlator function moments and of the Borel transform analysis (see Ref.[1]). The truncation of the set in the right hand side of eq.(25) leads to the mentioned unphysical dependence of the $f_{P,V}$ values on the external parameter of the sum rule scheme.

Further, let us use the conditions, simplifying the consideration due to the heavy quarkonium.

1.1 Nonperturbative Contribution

We assume that, in the limit of the very heavy quark mass, the power corrections of nonperturbative contribution are small. From eq.(19) one can see that, for example,

$$C_G^{(P)}(q^2) \approx O(\frac{1}{m_1 m_2}) , \quad \Lambda/m_{1,2} \ll 1 , \quad (26)$$

where v is fixed, $q^2 \sim (m_1 + m_2)^2$, when $\Im m \Pi^{(pert)}(q^2) \sim (m_1 + m_2)^2$. It is evident that, due to the purely dimensional consideration, one can believe that the Wilson's coefficients tend to zero as $1/m_{1,2}^2$.

Thus, the limit of very large heavy quark mass implies that one can neglect the quark-gluon condensate contribution.

1.2 Nonrelativistic Quark Motion

The nonrelativistic quark motion implies that, in the resonant region, one has, in accordance with eq.(20),

$$v \rightarrow 0 . \quad (27)$$

So, one can easily find that in the leading order

$$\Im m \Pi_P^{(pert)}(s) \approx \Im m \Pi_V^{(pert)}(s) \rightarrow \frac{3v}{8\pi^2} s \left(\frac{4\mu}{M} \right)^2 , \quad (28)$$

so that with account of the Coulomb factor

$$F(v) \simeq \frac{4\pi}{3} \frac{\alpha_S}{v} , \quad (29)$$

one obtains

$$\Im m \Pi_{P,V}^{(pert)}(s) \simeq \frac{\alpha_S}{2} s \left(\frac{4\mu}{M} \right)^2 . \quad (30)$$

1.3 "Smooth Average Value" Scheme of the Sum Rules

As for the hadronic part of the correlator, one can write down for the narrow resonance contribution

$$\Pi_V^{(res)}(q^2) = \int \frac{ds}{s - q^2} \sum_n f_{Vn}^2 M_{Vn}^2 \delta(s - M_{Vn}^2) , \quad (31)$$

$$\Pi_P^{(res)}(q^2) = \int \frac{ds}{s - q^2} \sum_n f_{Pn}^2 \frac{M_{Pn}^4}{(m_1 + m_2)^2} \delta(s - M_{Pn}^2) , \quad (32)$$

The integrals in eqs.(31)-(32) are simply calculated, and this procedure is generally used.

In the presented scheme, let us introduce the function of state number $n(s)$, so that

$$n(m_k^2) = k . \quad (33)$$

This definition seems to be reasonable in the resonant region. Then one has, for example, that

$$\frac{1}{\pi} \Im m \Pi_V^{(res)}(s) = s f_{Vn(s)}^2 \frac{d}{ds} \sum_k \theta(s - M_{V_k}^2) . \quad (34)$$

Further, it is evident that

$$\frac{d}{ds} \sum_k \theta(s - M_k^2) = \frac{dn(s)}{ds} \frac{d}{dn} \sum_k \theta(n - k) , \quad (35)$$

and eq.(31) can be rewritten as

$$\Pi_V^{(res)}(q^2) = \int \frac{ds}{s - q^2} s f_{Vn(s)}^2 \frac{dn(s)}{ds} \frac{d}{dn} \sum_k \theta(n - k) . \quad (36)$$

The "smooth average value" scheme means that

$$\Pi_V^{(res)}(q^2) = \langle \frac{d}{dn} \sum_k \theta(n - k) \rangle = \int \frac{ds}{s - q^2} s f_{Vn(s)}^2 \frac{dn(s)}{ds} . \quad (37)$$

It is evident that, in average, the first derivative of step-like function in the resonant region is equal to

$$\langle \frac{d}{dn} \sum_k \theta(n - k) \rangle \simeq 1 . \quad (38)$$

Thus, in the scheme one has

$$\langle \Pi_V^{(res)}(q^2) \rangle \approx \int \frac{ds}{s - q^2} s f_{Vn(s)}^2 \frac{dn(s)}{ds} , \quad (39)$$

$$\langle \Pi_P^{(res)}(q^2) \rangle \approx \int \frac{ds}{s - q^2} \frac{s^2 f_{Pn(s)}^2}{(m_1 + m_2)^2} \frac{dn(s)}{ds} . \quad (40)$$

Eqs.(39)-(40) give the average correlators for the vector and pseudoscalar mesons, therefore, due to eq.(25) we state that

$$\Im m \langle \Pi^{(hadr)}(q^2) \rangle = \Im m \Pi^{(QCD)}(q^2) , \quad (41)$$

that gives with account of eqs.(30), (39) and (40) at the physical points $s_n = M_n^2$

$$\frac{f_n^2}{M_n} = \frac{\alpha_S}{\pi} \frac{dM_n}{dn} \left(\frac{4\mu}{M} \right)^2, \quad (42)$$

where in the limit of heavy quarks we use, that for the resonances one has

$$m_1 + m_2 \approx M, \quad (43)$$

so that

$$f_{Vn} \simeq f_{Pn} = f_n. \quad (44)$$

Thus, one can conclude that for the heavy quarkonia the QCD sum rules give the identity of f_P and f_V values for the pseudoscalar and vector states.

Eq.(42) differs from the ordinary sum rule scheme because it does not contain the parameters, which are external to QCD. The quantity dM_n/dn is purely phenomenological. It defines the average mass difference between the nearest levels with the identical quantum numbers.

Further, as it has been shown in ref.[6], in the region of average distances between the heavy quarks in the charmonium and the bottomonium,

$$0.1 fm < r < 1 fm, \quad (45)$$

the QCD-motivated potentials allow the approximation in the form of logarithmic law [7] with the simple scaling properties, so

$$\frac{dn}{dM_n} = const., \quad (46)$$

i.e. the density of heavy quarkonium states with the given quantum numbers do not depend on the heavy quark flavours.

In ref.[4] it has been shown, that relation (46) is also practically valid for the heavy quark potential approximation by the power law (Martin potential) [8], where, neglecting a low value of the binding energy for the quarks inside the quarkonium, one can again get eq.(46).

In ref.[3] it has been found, that relation (46) is valid with the accuracy up to small logarithmic corrections over the reduced mass of quarkonium, if one makes the quantization of S -wave states for the quarkonium with the Martin potential by the Bohr-Sommerfeld procedure.

Moreover, with the accuracy up to the logarithmic corrections, α_S is the constant value. Thus, as it has been shown in refs.[3, 4], for the leptonic constants of S -wave quarkonia, the scaling relation takes place

$$\frac{f^2}{M} \left(\frac{M}{4\mu} \right)^2 = \text{const.} , \quad (47)$$

independently of the heavy quark flavours.

Taking into the account eqs.(43) and (44) and integrating eqs.(39), (40) by parts, one can get with the accuracy up to border terms, that one has

$$-2f_n \frac{df_n}{dn} \frac{dn}{dM_n} n = \frac{\alpha_s}{\pi} M_n \left(\frac{4\mu}{M_n} \right)^2 . \quad (48)$$

Comparing eqs.(42) and (48), one finds

$$\frac{df_n}{f_n dn} = -\frac{1}{2n} , \quad (49)$$

that gives, after the integration,

$$\frac{f_{n_1}^2}{f_{n_2}^2} = \frac{n_2}{n_1} . \quad (50)$$

Relation (50) leads to that the border terms, which have been neglected in the writing of eq.(48), are identically equal to zero.

First, note that eq.(47), relating the leptonic constants of different quarkonia, turns out to be certainly valid for the quarkonia with the hidden flavour ($c\bar{c}$, $b\bar{b}$), where $4\mu/M = 1$ (see [3, 4]).

Second, eq.(47) gives estimates of the leptonic constants for the heavy B and D mesons, so these estimates are in a good agreement with the values, obtained in the framework of other schemes of the QCD sum rules [4].

Third, taking a value of the $1S$ -level leptonic constant as the input one, we have calculated the leptonic constants of higher nS -excitations in the charmonium and the bottomonium and found a good agreement with the experimental values [5].

These three facts show that the offered scheme can be quite reliably applied to the systems with the heavy quarks.

Further, from eqs.(42) and (50) it follows that

$$\frac{f_1^2}{n} \frac{1}{M_n} = \frac{\alpha_S}{\pi} \left(\frac{4\mu}{M_n} \right)^2 \frac{dM_n}{dn}, \quad (51)$$

so that, neglecting the low value of quark binding energy ($M_n = M_1(1 + O(1/M))$), one gets

$$\frac{dM_n}{dn} = \frac{1}{n} \frac{dM_n}{dn} (n=1). \quad (52)$$

Integrating eq.(52), one partially finds eq.(3)

$$\frac{M_n - M_1}{M_2 - M_1} = \frac{\ln n}{\ln 2}, \quad n \geq 2, \quad (53)$$

and

$$M_2 - M_1 = \frac{dM_n}{dn} (n=1) \ln 2. \quad (54)$$

Thus, in the offered scheme of QCD sum rules, one takes into account the Coulomb-like α_S/v -corrections and, neglecting the power corrections over the inverse heavy quark mass, one gets the universal relation for the differences of nS -wave level masses of the heavy quarkonium.

2 Analysis of the Mass Difference Relation

Eq.(53) for the differences of nS -wave level masses of the heavy quarkonium does not contain external parameters and it allows direct comparison with the experimental data on the masses of particles in the ψ - and Υ -families [9].

Dependence (53) and the experimental values for the relations of heavy quarkonium masses are presented on Figure 1, where one neglects the spin-spin splittings.

Note, the $\psi(3770)$ and $\psi(4040)$ charmonium states suppose to be the results of the $3D$ - and $3S$ -states mixing, so that the D -wave dominates in the $\psi(3770)$ -state, and the mixing of the $3D$ and $3S$ wave functions is accompanied by a small shifts of the masses, so that we have supposed $M_3 = M_{\psi(4040)}$.

As one can see from the Figure, relation (53), obtained in the leading approximation, is in a good agreement with the experimental data¹.

¹Note, that at $n \geq 4$, the nS -level is above the threshold of decay into the heavy meson pair and the dynamics of light quarks becomes essential. This dynamics is not taken into account in the present model.

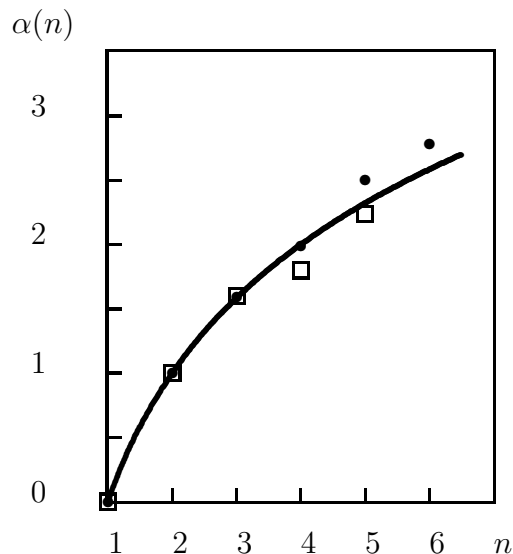


Figure 1: The experimental values of nS - bottomonium (solid dots) and charmonium (empty boxes) mass differences $\alpha(n) = (M_n - M_1)/(M_2 - M_1)$ and the dependence in the present model $\alpha(n) = \ln n / \ln 2$.

Conclusion

In the framework of the specific scheme of QCD sum rules for the two-point correlators of heavy quark currents, in the leading approximation one takes into account the contribution of higher nS -levels in the heavy quarkonium and derives the universal relation for the quarkonium mass differences

$$\frac{M_n - M_1}{M_2 - M_1} = \frac{\ln n}{\ln 2}, \quad n \geq 2.$$

This reflects the phenomenological flavour independence of the kinetic energy in the bound states of heavy quarkonium.

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